

Μεθοδικά, απλά & κατανοητά...

Θέμα

Έστω \hat{A} , \hat{B} και \hat{C} γραμμικοί τελεστές. Δείξτε ότι

$$(a) [\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$$

$$(b) [\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

$$(c) [\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0$$

(d) Χρησιμοποιώντας τα παραπάνω υπολογίστε τον μεταθέτη $[\hat{x}, \hat{p}^n]$ όπου n θετικός ακέραιος.

$$a) \text{ Είναι: } [\hat{A}, \hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \overset{\infty}{=} \text{προσθαφαιρούμε το όρο } \hat{B}\hat{A}\hat{C}$$

$$= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{A}\hat{C} =$$

$$= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} =$$

$$= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A}) =$$

$$= [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$$

$$b) \text{ Είναι: } [\hat{A}\hat{B}, \hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = \overset{\infty}{=} \text{προσθαφαιρούμε το όρο } \hat{A}\hat{C}\hat{B}$$

$$= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{A}\hat{C}\hat{B} =$$

$$= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} =$$

$$= \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + (\hat{A}\hat{C} - \hat{C}\hat{A})\hat{B} =$$

$$= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

Μεθοδικά, απλά & κατανοητά...

$$\begin{aligned}
 c) & [\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = \\
 & = [\hat{A}, \hat{B}\hat{C} - \hat{C}\hat{B}] + [\hat{B}, \hat{C}\hat{A} - \hat{A}\hat{C}] + [\hat{C}, \hat{A}\hat{B} - \hat{B}\hat{A}] = \\
 & = [\hat{A}, \hat{B}\hat{C}] - [\hat{A}, \hat{C}\hat{B}] + [\hat{B}, \hat{C}\hat{A}] - [\hat{B}, \hat{A}\hat{C}] + [\hat{C}, \hat{A}\hat{B}] - [\hat{C}, \hat{B}\hat{A}] = \\
 & = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} - (\hat{A}\hat{C}\hat{B} - \hat{C}\hat{B}\hat{A}) + \hat{B}\hat{C}\hat{A} - \hat{C}\hat{A}\hat{B} - (\hat{B}\hat{A}\hat{C} - \hat{A}\hat{C}\hat{B}) + \\
 & + \hat{C}\hat{A}\hat{B} - \hat{A}\hat{B}\hat{C} - (\hat{C}\hat{B}\hat{A} - \hat{B}\hat{A}\hat{C}) = \\
 & = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} - \hat{A}\hat{C}\hat{B} + \hat{C}\hat{B}\hat{A} + \hat{B}\hat{C}\hat{A} - \hat{C}\hat{A}\hat{B} - \hat{B}\hat{A}\hat{C} + \hat{A}\hat{C}\hat{B} + \\
 & + \hat{C}\hat{A}\hat{B} - \hat{A}\hat{B}\hat{C} - \hat{C}\hat{B}\hat{A} + \hat{B}\hat{A}\hat{C} = 0
 \end{aligned}$$

d) Είναι: $[\hat{x}, \hat{p}^2] = [\hat{x}, \hat{p}\hat{p}] = [\hat{x}, \hat{p}]\hat{p} + \hat{p}[\hat{x}, \hat{p}] = 2i\hbar\hat{p}$

$[\hat{x}, \hat{p}^3] = [\hat{x}, \hat{p}\hat{p}^2] = [\hat{x}, \hat{p}]\hat{p}^2 + \hat{p}[\hat{x}, \hat{p}^2] = 3i\hbar\hat{p}^2$

$[\hat{x}, \hat{p}^4] = [\hat{x}, \hat{p}\hat{p}^3] = [\hat{x}, \hat{p}]\hat{p}^3 + \hat{p}[\hat{x}, \hat{p}^3] = 4i\hbar\hat{p}^3$

⋮ (αναγωγή)

$[\hat{x}, \hat{p}^{n-2}] = [\hat{x}, \hat{p}\hat{p}^{n-2}] = [\hat{x}, \hat{p}]\hat{p}^{n-2} + \hat{p}[\hat{x}, \hat{p}^{n-2}] = (n-2)i\hbar\hat{p}^{n-2}$

$\Rightarrow [\hat{x}, \hat{p}^n] = [\hat{x}, \hat{p}\hat{p}^{n-1}] = [\hat{x}, \hat{p}]\hat{p}^{n-1} + \hat{p}[\hat{x}, \hat{p}^{n-1}] = ni\hbar\hat{p}^{n-1}$

Άρα: $[\hat{x}, \hat{p}^n] = ni\hbar\hat{p}^{n-1}$

