Άσκηση 6

$$\Psi = \underbrace{\left(\frac{1-i}{2}\right)}_{C_1}\psi_1 + \underbrace{N_2}_{C_2}\psi_2$$

$$P_2 = |N_2|^2$$

Συνθήκη κανονικοποίησης: $|C_1|^2 + |C_2|^2 = 1 \rightarrow$

$$\rightarrow P_1 + P_2 = 1 \rightarrow P_1 = 1 - P_2 \quad (1)$$

$$\langle \hat{A} \rangle = |C_1|^2\lambda_1 + |C_2|^2\lambda_2 = P_1\lambda_1 + P_2\lambda_2 \overset{(1)}{\Longrightarrow}$$

$$\rightarrow \boxed{\langle \hat{A} \rangle = (1-P_2)\lambda_1 + P_2\lambda_2} \quad (2)$$

$$\langle \hat{A}^2 \rangle = |C_1|^2\lambda_1^2 + |C_2|^2\lambda_2^2 = P_1\lambda_1^2 + P_2\lambda_2^2 \overset{(1)}{\Longrightarrow}$$

$$\rightarrow \langle \hat{A}^2 \rangle = (1-P_2)\lambda_1^2 + P_2\lambda_2^2 \quad (3)$$

Άρα: $(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \overset{(2,3)}{=} (1-P_2)\lambda_1^2 + P_2\lambda_2^2 - \left[(1-P_2)\lambda_1 + P_2\lambda_2\right]^2 =$

$$= (1-P_2)\lambda_1^2 + P_2\lambda_2^2 - (1-P_2)^2\lambda_1^2 - P_2^2\lambda_2^2 - 2P_2(1-P_2)\lambda_1\lambda_2$$

$$\rightarrow \boxed{\Delta A = \sqrt{(1-P_2)\lambda_1^2(\cancel{1}-\cancel{1}+P_2) + P_2\lambda_2^2(1-P_2) - 2P_2(1-P_2)\lambda_1\lambda_2}} \quad (4)$$

Αλλά: $P_1 = |C_1|^2 = \left|\frac{1-i}{2}\right|^2 = \left(\frac{1-i}{2}\right)\left(\frac{1-i}{2}\right)^* = \left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right) =$

$$= \frac{(1-i)(1+i)}{4} = \frac{1^2 - i^2}{4} =$$

$|z|^2 = z z^*$

$$= \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2}.$$

Σ $(1) \rightsquigarrow P_2 = 1 - P_1 = 1 - \frac{1}{2} \rightarrow \boxed{P_2 = \frac{1}{2}}$

Άρα: $(4) \rightsquigarrow \Delta A = \sqrt{\frac{1}{4}\lambda_1^2 + \frac{1}{4}\lambda_2^2 - 2\frac{1}{4}\lambda_1\lambda_2} =$

$$= \frac{1}{2}\sqrt{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2} =$$

$$= \frac{1}{2}\sqrt{(\lambda_1 - \lambda_2)^2} \rightarrow \boxed{\Delta A = \frac{|\lambda_1 - \lambda_2|}{2}}$$

ΚΥΜΑΤΟΣΥΝΑΡΤΗΣΗ ΟΡΜΗΣ

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\,\psi(x)\,dx$$

Άσκηση 1.

$$\psi(x) = \begin{cases} 0, & x<0 \\ N\sin\left(\frac{4\pi}{L}x\right), & 0 \le x \le L \\ 0, & x>L \end{cases}$$

α) $N = ?$ Συνθήκη κανονικοποίησης:

$$\int_{-\infty}^{+\infty} \psi^* \psi\, dx = 1 \rightarrow \int_0^L N^* \sin\left(\frac{4\pi}{L}x\right) N \sin\left(\frac{4\pi}{L}x\right) dx = 1 \rightarrow$$

$$\rightarrow |N|^2 \int_0^L \sin^2\left(\frac{4\pi}{L}x\right) dx = |N|^2 \frac{1}{2} \int_0^L \left(1-\cos\left(\frac{8\pi x}{L}\right)\right) dx =$$

$$\sin^2\left(\frac{4\pi}{L}x\right) = \frac{1}{2}\left(1-\cos\frac{8\pi x}{L}\right)$$

$$= \frac{|N|^2}{2}\left[x - \frac{\sin\left(\frac{8\pi x}{L}\right)}{\frac{8\pi}{L}}\right]_0^L = \frac{|N|^2}{2}\left(L - \frac{\sin 8\pi - \sin 0}{\frac{8\pi}{L}}\right) =$$

$$= \frac{|N|^2}{2} L = 1 \rightarrow |N|^2 = \frac{2}{L} \rightarrow \boxed{N = \sqrt{\frac{2}{L}}}$$

Κανονικοποιημένη κυματοσυνάρτηση: $\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{4\pi}{L}x\right)$ (1)

Σχέση ορθογωνιότητας: $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{nm}, \quad \delta_{nm} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$

$$\sin\vartheta = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$$

β) $\varphi(p) = ?$

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\psi(x)\,dx \overset{(1)}{=} \frac{1}{\sqrt{2\pi\hbar}} \int_0^L e^{-ipx/\hbar}\sqrt{\frac{2}{L}}\sin\left(\frac{4\pi}{L}x\right) dx =$$

$$= \frac{1}{\sqrt{\pi\hbar L}} \int_0^L e^{-ipx/\hbar} \sin\left(\frac{4\pi}{L}x\right) dx = \frac{1}{2i\sqrt{\pi\hbar L}} \int_0^L \left(e^{i\left(\frac{4\pi}{L}-\frac{p}{\hbar}\right)x} - e^{-i\left(\frac{4\pi}{L}+\frac{p}{\hbar}\right)x}\right) dx =$$

$$\sin\left(\frac{4\pi}{L}x\right) = \frac{e^{i\frac{4\pi}{L}x} - e^{-i\frac{4\pi}{L}x}}{2i}$$

$$= \frac{1}{2i\sqrt{\pi\hbar L}}\left[\frac{e^{i\left(\frac{4\pi}{L}-\frac{p}{\hbar}\right)x}}{i\left(\frac{4\pi}{L}-\frac{p}{\hbar}\right)} - \frac{e^{-i\left(\frac{4\pi}{L}+\frac{p}{\hbar}\right)x}}{-i\left(\frac{4\pi}{L}+\frac{p}{\hbar}\right)}\right]_0^L = \frac{1}{2i^2\sqrt{\pi\hbar L}}\left[\frac{e^{i\left(4\pi-\frac{pL}{\hbar}\right)} - \cancel{e^0}^{\,1}}{\frac{4\pi}{L}-\frac{p}{\hbar}} + \frac{e^{-i\left(4\pi+\frac{pL}{\hbar}\right)} - \cancel{e^0}^{\,1}}{\frac{4\pi}{L}+\frac{p}{\hbar}}\right]$$

$$= \frac{-1}{2\sqrt{\pi\hbar L}}\left[\frac{\cancel{e^{i4\pi}}^{\,1} \cdot e^{-i\frac{pL}{\hbar}} - 1}{\frac{4\pi}{L}-\frac{p}{\hbar}} + \frac{\cancel{e^{-i4\pi}}^{\,1}\, e^{-i\frac{pL}{\hbar}} - 1}{\frac{4\pi}{L}+\frac{p}{\hbar}}\right] =$$

$$e^{\pm i4\pi} = \cos 4\pi \pm i\sin 4\pi = 1$$

Τύπος Euler: $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$

$$= -\frac{\left(e^{-i\frac{pL}{\hbar}} - 1\right)}{2\sqrt{\pi\hbar L}}\left[\frac{\frac{4\pi}{L} + \cancel{\frac{p}{\hbar}} + \frac{4\pi}{L} - \cancel{\frac{p}{\hbar}}}{\left(\frac{4\pi}{L}-\frac{p}{\hbar}\right)\left(\frac{4\pi}{L}+\frac{p}{\hbar}\right)}\right] = \frac{1-e^{-i\frac{pL}{\hbar}}}{\cancel{2}\sqrt{\pi\hbar L}} \frac{\frac{4\cancel{2}\pi}{L}}{\frac{16\pi^2}{L^2} - \frac{p^2}{\hbar^2}} \rightarrow$$

$$\rightarrow \boxed{\varphi(p) = \frac{4\pi}{L\sqrt{\pi\hbar L}} \frac{1-e^{-i\frac{pL}{\hbar}}}{\frac{16\pi^2}{L^2} - \frac{p^2}{\hbar^2}}}$$

$$\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\,\psi(x)\,dx$$

Άσκηση 1.

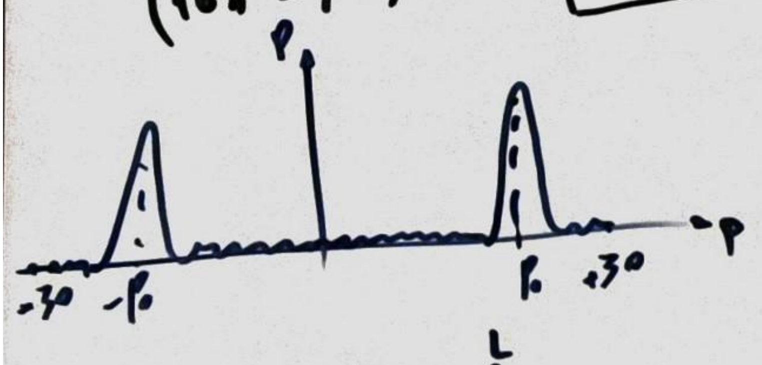
γ) $P(p) = |\varphi(p)|^2 = \varphi^*(p)\,\varphi(p)$ $\quad \underline{(\hbar = L = 1)}$

$$= \frac{4\pi}{\sqrt{\pi}}\,\frac{1-e^{+ip}}{16\pi^2-p^2}\cdot\frac{4\pi}{\sqrt{\pi}}\,\frac{1-e^{-ip}}{16\pi^2-p^2} =$$

$$= \frac{16\,\pi\cancel{\pi}\,(1-e^{ip})(1-e^{-ip})}{\cancel{\pi}\,(16\pi^2-p^2)^2} =$$

$$= \frac{16\pi\,(1-e^{-ip}-e^{ip}+\cancel{e^{ip}e^{-ip}})}{(16\pi^2-p^2)^2} = \frac{16\pi\left[2-\overbrace{(e^{-ip}+e^{ip})}^{2\cos p}\right]}{(16\pi^2-p^2)^2} =$$

$$= \frac{16\pi\,(2-2\cos p)}{(16\pi^2-p^2)^2} \rightarrow \boxed{|\varphi(p)|^2 = \frac{32\pi\,(1-\cos p)}{(16\pi^2-p^2)^2}}$$



$$= \frac{1}{\sqrt{\pi\hbar L}} \int_0^L e^{-ipx/\hbar}\,\sin\left(\frac{4\pi}{L}x\right) \ldots \qquad \frac{e^{i\frac{4\pi}{L}x}-e^{-i\ldots}}{2i}$$

$$= \frac{1}{2i\sqrt{\pi\hbar L}}\left[\frac{e^{i\left(\frac{4\pi}{L}-\frac{p}{\hbar}\right)x}}{i\left(\frac{4\pi}{L}-\frac{p}{\hbar}\right)} - \ldots\right.$$

$$= \frac{-1}{2\sqrt{\pi\hbar L}}\left[\frac{\cancel{e^{i4\pi}}\ldots}{\frac{4\pi}{L}\ldots}\right.$$

$$= -\frac{\left(e^{-i\frac{pL}{\hbar}}-1\right)}{2\sqrt{\pi\hbar L}}\left[\ldots\right.$$

$$\rightarrow |\varphi(\ldots$$

Σχέση ορθογωνιότητας: $\displaystyle\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{nm}$, $\quad \delta_{nm} = \begin{cases} 0, & n\neq m \\ 1, & n=m \end{cases}$

$$\sin\vartheta = \frac{e^{i\vartheta}-e^{-i\vartheta}}{2i}, \quad \cos\vartheta = \frac{e^{i\vartheta}+e^{-i\vartheta}}{2}$$

ΡΕΥΜΑ ΠΥΚΝΟΤΗΤΑΣ ΠΙΘΑΝΟΤΗΤΑΣ J

εξίσωση συνέχειας: $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$

όπου $\rho = |\psi|^2 = \psi\psi^*$ πυκνότητα πιθανότητας

και $J_{(x)} = \frac{i\hbar}{2m}\left[\psi(x)\frac{\partial \psi^*}{\partial x} - \psi^*\frac{\partial \psi}{\partial x}\right]$

Άσκηση 1:

α) $\psi(x) = Ae^{ikx} + Be^{-ikx}$, $A, B \in \mathbb{C}$, $k \in \mathbb{R}$.

$\psi^* = A^*e^{-ikx} + B^*e^{ikx}$

$\frac{\partial \psi}{\partial x} = iAke^{ikx} - iBke^{-ikx}$

$\frac{\partial \psi^*}{\partial x} = -iA^*ke^{-ikx} + iB^*ke^{ikx}$

Άρα: $J = \frac{i\hbar}{2m}\left[\psi\frac{\partial \psi^*}{\partial x} - \psi^*\frac{\partial \psi}{\partial x}\right] =$

$$= \frac{i\hbar}{2m}\Big[(Ae^{ikx} + Be^{-ikx})(-iA^*ke^{-ikx} + iB^*ke^{ikx}) - (A^*e^{-ikx} + B^*e^{ikx})(iAke^{ikx} - iBke^{-ikx})\Big] =$$

$$= \frac{i\hbar}{2m}\Big(-ikAA^* + \cancel{ikAB^*e^{2ikx}} - \cancel{ikBA^*e^{-2ikx}} + ikBB^* - ikA^*A + \cancel{ikA^*Be^{-2ikx}} - \cancel{ikB^*Ae^{2ikx}} + ikB^*B\Big) =$$

$$= \frac{i^2\hbar k}{\cancel{2}m}\left(-\cancel{2}A^*A + \cancel{2}B^*B\right) = -\frac{\hbar k}{m}\left(-|A|^2 + |B|^2\right) \rightarrow \boxed{J = \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right)}$$

β) $\psi(x) = Ae^{-ax}$ $A \in \mathbb{C}$, $a \in \mathbb{R}$, $0 < x < \infty$.

$\psi^* = A^*e^{-ax}$, $\frac{\partial \psi}{\partial x} = -Aae^{-ax}$, $\frac{\partial \psi^*}{\partial x} = -A^*ae^{-ax}$

Άρα: $J = \frac{i\hbar}{2m}\left[\psi\frac{\partial \psi^*}{\partial x} - \psi^*\frac{\partial \psi}{\partial x}\right] = \frac{i\hbar}{2m}\left[Ae^{-ax}(-A^*ae^{-ax}) - A^*e^{-ax}(-Aae^{-ax})\right]$

$$= \frac{i\hbar}{2m}\left[\cancel{-AA^*ae^{-2ax} + A^*Aae^{-2ax}}\right] \rightarrow \boxed{J = 0}$$

γ) $\psi(x) = R(x)e^{iS(x)/\hbar}$, $R(x), S(x) \in \mathbb{R}$.

$\psi^* = R(x)e^{-iS/\hbar}$, $\frac{\partial \psi}{\partial x} = R'e^{iS/\hbar} + Re^{iS/\hbar}\cdot\frac{iS'}{\hbar} = \left(R' + i\frac{RS'}{\hbar}\right)e^{iS/\hbar}$

$\frac{\partial \psi^*}{\partial x} = \left(R' - i\frac{RS'}{\hbar}\right)e^{-iS/\hbar}$.

Άρα: $J = \frac{i\hbar}{2m}\left[Re^{iS/\hbar}\left(R' - i\frac{RS'}{\hbar}\right)e^{-iS/\hbar} - Re^{-iS/\hbar}\left(R' + i\frac{RS'}{\hbar}\right)e^{iS/\hbar}\right] =$

$$= \frac{i\hbar}{2m}R\left(\cancel{R'} - i\frac{RS'}{\hbar} - \cancel{R'} - i\frac{RS'}{\hbar}\right) = \frac{-i^2\cancel{\hbar}R^2\cancel{2}S'}{\cancel{2}m\cancel{\hbar}} \rightarrow \boxed{J = \frac{R^2_{(x)}}{m}S'_{(x)}}$$

Ηλεκτροδυναμική.

εξισ. συνέχειας: $\underset{\text{πυκνότητα ρεύματος}}{\vec{\nabla}\cdot\vec{J}} + \frac{\partial \rho}{\partial t} = 0$ (ρ: πυκνότητα φορτίου)

Α.Δ. Φορτίου

Ρευστομηχανική:

εξισ. συνέχειας: $\vec{\nabla}\cdot(\overset{J}{\rho\vec{u}}) + \frac{\partial \rho}{\partial t} = 0$

(Α.Δ. Μάζας)

Άσκηση 2.

ν.δ.ο. ⇝ $\frac{dP_{ab}(t)}{dt} = J(a,t) - J(b,t)$

Πυκνότητα πιθανότητας: $P(x,t) = |\Psi(x,t)|^2$ οπότε η πιθανότητα να βρίσκεται το σωματίδιο στην περιοχή $a<x<b$ είναι:

$$P_{ab} = \int_a^b P(x,t)\,dx \quad (1)$$

Από εξίσωση συνέχειας: $\vec{\nabla}\cdot\vec{J} + \frac{\partial P(x,t)}{\partial t} = 0$

Λόγω μονοδιάστατου προβλήματος είναι: $\vec{\nabla}\cdot\vec{J} = \frac{\partial J(x,t)}{\partial x}$

$$\rightarrow \frac{\partial J(x,t)}{\partial x} + \frac{\partial P(x,t)}{\partial t} = 0$$

$$\rightarrow \frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \xrightarrow[\text{από } a \text{ ως } b]{\text{ολοκληρώνω}} \int_a^b \frac{\partial P(x,t)}{\partial t}\,dx = -\int_a^b \frac{\partial J(x,t)}{\partial x}\,dx \rightarrow$$

$$\rightarrow \frac{\partial}{\partial t}\int_a^b P(x,t)\,dx = -\int_a^b \partial J(x,t) \xrightarrow{(1)} \frac{\partial P_{ab}}{\partial t} = -J(x,t)\Big|_a^b \rightarrow$$

$$\rightarrow \frac{\partial P_{ab}}{\partial t} = -J(b,t) + J(a,t) \rightarrow \boxed{\frac{\partial P_{ab}}{\partial t} = J(a,t) - J(b,t)}$$

$$\vec{\nabla}\cdot\vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}.$$

ΕΞΙΣΩΣΗ ΙΔΙΟΤΙΜΩΝ ΕΝΕΡΓΕΙΑΣ (HAMILTONIAN)

$$\boxed{\hat{H}\psi_n(x) = E_n \psi_n(x)} \quad (1)$$

συμπαγής μορφή της χρονοανεξάρτητης εξίσ. Schrödinger

ιδιοσυνάρτηση της Hamiltonian — ιδιοτιμές ενέργειας

Επειδή $\left.\begin{aligned}\hat{H} &= \frac{\hat{p}^2}{2m} + V(x) \\ \hat{p}^2 &= -\hbar^2 \frac{\partial^2}{\partial x^2}\end{aligned}\right\} \rightarrow \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \quad (2)$

Θέτω η (1) λόγω της (2) δίνει:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi_n = E_n \psi_n \rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2 \psi_n}{\partial x^2} + V(x)\psi_n = E\psi_n \rightarrow$$

$$\rightarrow \boxed{\frac{\hbar^2}{2m}\frac{\partial^2 \psi_n}{\partial x^2} + (E - V(x))\psi_n = 0} \quad \text{ή} \quad \boxed{\frac{\partial^2 \psi_n}{\partial x^2} + \frac{2m}{\hbar^2}(E - V(x))\psi_n = 0}$$

Γεν. λύση: $\psi_n(x)$.

$\psi(x,t) = \psi(x)\,U(t) \quad (3)$

Χρονοεξαρτώμενη εξίσ. Schrödinger: $\hat{H}\psi(x,t) = i\hbar \frac{\partial \psi(x,t)}{\partial t} \xrightarrow{(3)}$

$$\rightarrow \hat{H}\cancel{\psi(x)}U(t) = i\hbar\,\cancel{\psi(x)}\frac{\partial U(t)}{\partial t} \rightarrow \frac{\partial U(t)}{\partial t} = \frac{1}{i\hbar}\hat{H}U(t) = \frac{i}{i^2\hbar}\hat{H}U(t) \rightarrow$$

$$\rightarrow \frac{\partial U}{\partial t} = -\frac{i}{\hbar}\hat{H}U \rightarrow \int \frac{\partial U}{U} = -\frac{i}{\hbar}\int \hat{H}\,\partial t \rightarrow \ln U = -\frac{i}{\hbar}\hat{H}t \rightarrow$$

$$\rightarrow \boxed{U(t) = e^{-\frac{i}{\hbar}\hat{H}t}}$$

Άρα: $\psi(x,t) = \psi(x)\,e^{-\frac{i}{\hbar}\hat{H}t}$.

Γενικά: $\psi(x,t) = \sum_n c_n \psi_n(x)\, e^{-i\frac{E_n t}{\hbar}}$

Τελεστής χρονικής εξέλιξης $\hat{U}(t)$.

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$$

$$\psi(x,t) = \psi(x,t_0)\,\hat{U}(t) = \psi(x,t_0)\,e^{-\frac{i\hat{H}}{\hbar}t}$$

π.χ. $\psi(x,t_0=0) = c_1\psi_1 + c_2\psi_2 + \dots + c_n\psi_n$

τότε: $\psi(x,t) = \psi(x,0)\,\hat{U}(t) = \psi(x,0)\,e^{-i\frac{\hat{H}}{\hbar}t} \rightarrow$

$$\rightarrow \psi(x,t) = c_1 e^{-\frac{i E_1 t}{\hbar}}\psi_1 + c_2 e^{-\frac{i E_2 t}{\hbar}}\psi_2 + \dots + c_n e^{-i\frac{E_n}{\hbar}t}\psi_n.$$

Άσκηση 1

$$\psi(x,t=0) = N(\psi_1 + 2\psi_2)$$

α) ψ_1, ψ_2 : κανονικοποιημένες ιδιοσυναρτήσεις ενέργειας με ιδιοτιμές E_1 και E_2.

$\langle E \rangle = ?$ $\Delta E = ?$

Συνθήκη κανονικοποίησης: $\sum_n |c_n|^2 = 1 \rightarrow |c_1|^2 + |c_2|^2 = 1 \rightarrow$

$\rightarrow |N|^2 + |2N|^2 = 1 \rightarrow 5|N|^2 = 1 \rightarrow \boxed{N = \frac{1}{\sqrt{5}}}$

δηλ. $\psi(x,t=0) = \frac{1}{\sqrt{5}}\psi_1 + \frac{2}{\sqrt{5}}\psi_2$.

$$\langle E \rangle = \sum_n |c_n|^2 E_n = |c_1|^2 E_1 + |c_2|^2 E_2 =$$

$$= \left|\frac{1}{\sqrt{5}}\right|^2 E_1 + \left|\frac{2}{\sqrt{5}}\right|^2 E_2 \rightarrow \boxed{\langle E \rangle = \frac{E_1 + 4E_2}{5}} \quad (1)$$

$$\langle E^2 \rangle = \sum_n |c_n|^2 E_n^2 = |c_1|^2 E_1^2 + |c_2|^2 E_2^2 = \frac{E_1^2 + 4E_2^2}{5} \quad (2)$$

Άρα: $(\Delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2 \overset{(1),(2)}{=}$

$$= \frac{E_1^2 + 4E_2^2}{5} - \left(\frac{E_1 + 4E_2}{5}\right)^2 =$$

$$= \frac{E_1^2 + 4E_2^2}{5} - \frac{E_1^2 + 16E_2^2 + 8E_1E_2}{25} =$$

$$= \frac{5E_1^2 + 20E_2 - E_1^2 - 16E_2^2 - 8E_1E_2}{25} =$$

$$= \frac{4E_1^2 + 4E_2^2 - 8E_1E_2}{25} = \frac{4}{25}(E_1^2 + E_2^2 - 2E_1E_2) \rightarrow$$

$$\rightarrow (\Delta E)^2 = \frac{4}{25}(E_1 - E_2)^2 \rightarrow \boxed{\Delta E = \frac{2}{5}|E_1 - E_2|}$$