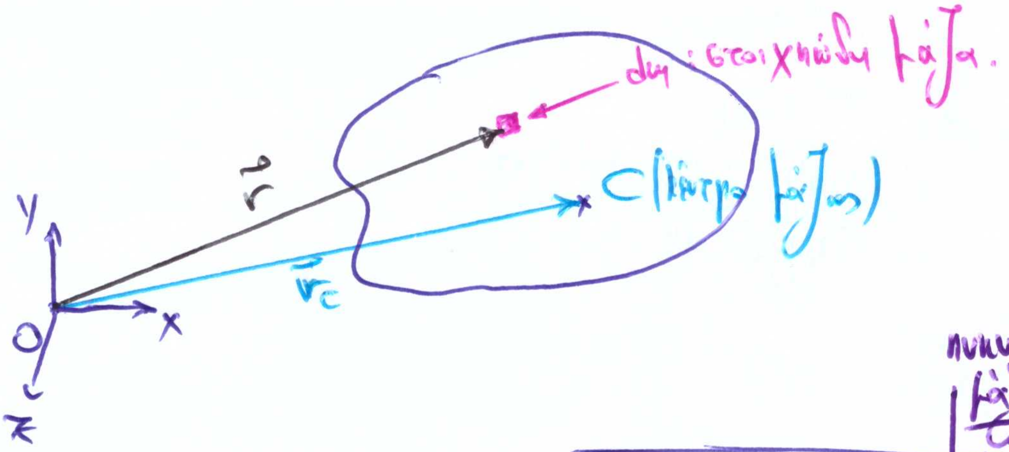
ΚΕΝΤΡΟ ΜΑΖΑΣ ΣΤΕΡΕΩΝ ΣΩΜΑΤΩΝ



Διάνυσμα θέσης κ.μ.:

πυκνότητα μάζας

$$\boxed{\vec{r}_c = \frac{1}{M}\int \vec{r}\,dm}$$

ολική μάζα

$$x_c = \frac{1}{M}\int x\,dm$$

$$y_c = \frac{1}{M}\int y\,dm$$

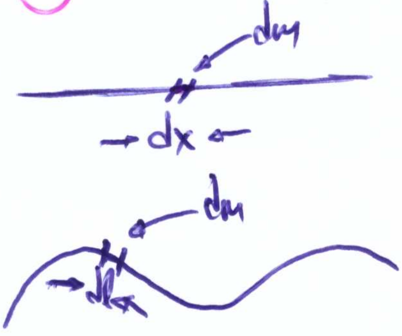
$$z_c = \frac{1}{M}\int z\,dm$$

Πυκνότητα μάζας

Ορισμοί

Για ομογενή σώματα είναι πυκνότητα σταθερή

1) Γραμμική (1-Δ):



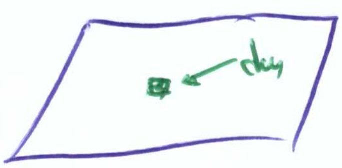
$$\boxed{\lambda = \frac{dm}{dx}}$$

$$\boxed{\lambda = \frac{dm}{dl}}$$

$$\boxed{\lambda = \frac{M}{l}}$$

Απόδειξη: $\lambda = \frac{dm}{dx} \to \int_0^M dm = \lambda \int_0^l dx \to M = \lambda l \to \lambda = \frac{M}{l}$

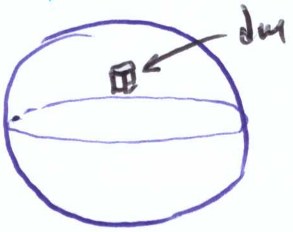
2) Επιφανειακή (2-Δ):



$$\boxed{\sigma = \frac{dm}{dS}}$$

$$\boxed{\sigma = \frac{M}{S}}$$

3) Χωρική (3-Δ):

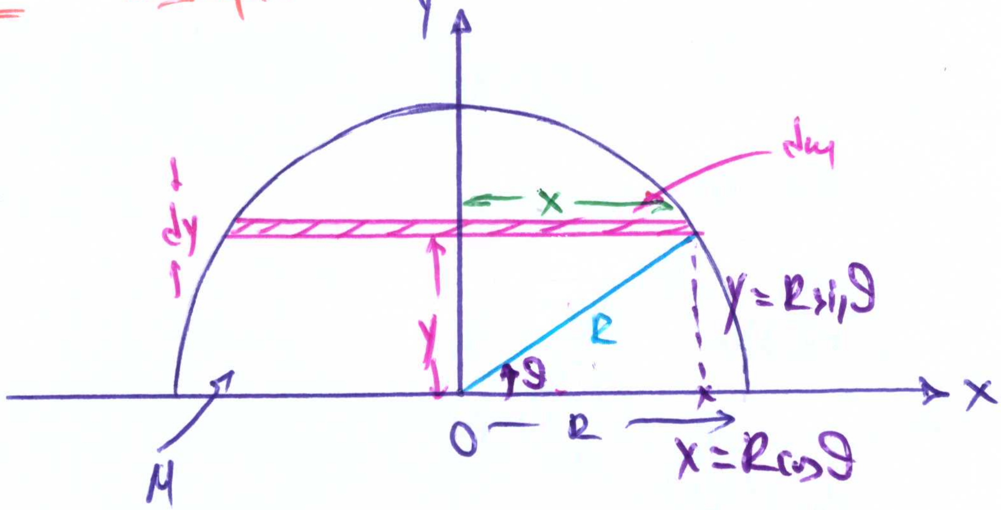


$$\boxed{\rho = \frac{dm}{dV}}$$

$$\boxed{\rho = \frac{M}{V}}$$

Για μη ομογενή σώματα η πυκνότητα είναι μεταβλητή δηλ.
$\lambda = \lambda(x)$, $\rho = \rho(r)$ κ.τ.λ. και η συνάρτηση αυτή θα δίνεται στην εκφώνηση.

Άσκηση 4.3 Ημιδίσκος



Λόγω συμμετρίας το κ.μ. θα βρίσκεται πάνω στο άξονα y δηλ. $x_c = 0$. Οπότε: $y_c = \frac{1}{M}\int y\,dm$ (1)

Αλλά: $\sigma = \frac{dm}{dS} \to dm = \sigma\, dS = \sigma\, 2x\, dy$

Επειδή η επιφάνεια είναι ομογενής: $\sigma = \frac{M}{S} = \frac{M}{\frac{\pi R^2}{2}} = \frac{2M}{\pi R^2}$ $\to$

$\to dm = \frac{2M}{\pi R^2}\, 2x\, dy \to dm = \frac{4M}{\pi R^2}\, x\, dy$ (2)

Οπότε: (1) $\overset{(2)}{\rightsquigarrow}$ $y_c = \frac{1}{\cancel{M}}\frac{4\cancel{M}}{\pi R^2}\int y\, x\, dy$ (3)

Αλλά: $x = R\cos\vartheta$ και $y = R\sin\vartheta \to dy = R\cos\vartheta\, d\vartheta$ (4)

Άρα: (3) $\overset{(4)}{\rightsquigarrow}$ $y_c = \frac{4}{\pi \cancel{R^2}}\int R\sin\vartheta\, \cancel{R}\cos\vartheta\, \cancel{R}\cos\vartheta\, d\vartheta =$

$= \frac{4R}{\pi}\int_{\vartheta=0}^{\pi/2} \cos^2\vartheta\, \sin\vartheta\, d\vartheta$ (5)

Θέτω : $u = \cos\theta \rightarrow du = -\sin\theta \, d\theta \rightarrow \sin\theta \, d\theta = -du$

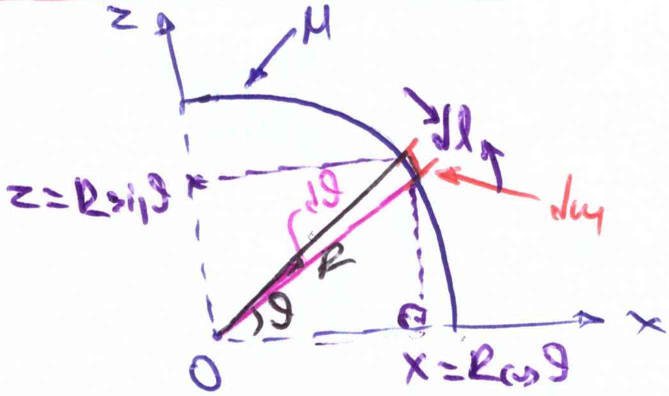
οπότε το |7| γίνεται:

$$y_c = \frac{4R}{\pi} \int u^2 (-du) = -\frac{4R}{\pi} \int u^2 du = -\frac{4R}{\pi} \frac{u^3}{3} =$$

$$= -\frac{4R}{3\pi} \cos^3\theta \Big|_0^{\pi/2} = -\frac{4R}{3\pi} \left(\cancelto{0}{\cos^3 \frac{\pi}{2}} - \cancelto{1}{\cos^3 0} \right) \rightarrow$$

$$\rightarrow \boxed{y_c = \frac{4R}{3\pi}}$$

Άσκηση 4.4. → Άσκηση 2 27/7/2019



Οπότε: $x_c = \frac{1}{M}\int x\,dm$ (1)

$z_c = \frac{1}{M}\int z\,dm$ (2)

Αλλά: $\lambda = \frac{dm}{dl} \to dm = \lambda\, dl$

Λόγω ομοιογένειας: $\lambda = \frac{M}{\frac{\pi R}{2}} = \frac{2M}{\pi R}$

$\to dm = \frac{2M}{\pi R}\,dl$ (3)

Οπότε: $(1) \overset{(3)}{\rightsquigarrow} x_c = \frac{1}{\cancel{M}}\,\frac{2\cancel{M}}{\pi R}\int x\,dl \to x_c = \frac{2}{\pi R}\int x\,dl$ (4)

$(2) \overset{(3)}{\rightsquigarrow} z_c = \frac{1}{\cancel{M}}\,\frac{2\cancel{M}}{\pi R}\int z\,dl \to z_c = \frac{2}{\pi R}\int z\,dl$ (5)

Αλλά: $x = R\cos\theta$, $z = R\sin\theta$ και $dl = R\,d\theta$

$\frac{2\pi}{\theta} = \frac{2\pi R}{l}$ | $l = \frac{\cancel{2\pi} R\theta}{\cancel{2\pi}}$

$\to \boxed{l = R\theta}$

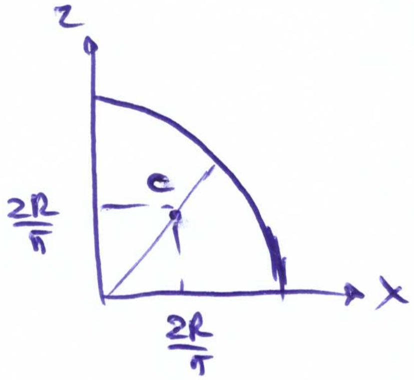
οπότε:

$(4) \rightsquigarrow x_c = \frac{2}{\pi \cancel{R}}\int_0^{\pi/2} R\cos\theta\, \cancel{R}\, d\theta = \frac{2R}{\pi}\int_0^{\pi/2}\cos\theta\, d\theta =$

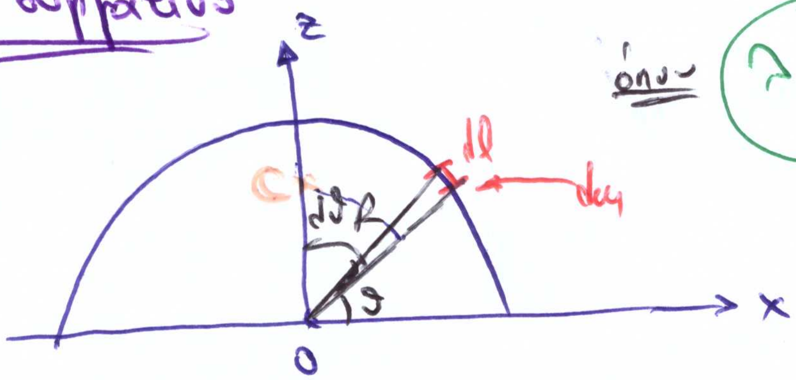
$= \frac{2R}{\pi}\left[\sin\theta\right]_0^{\pi/2} = \frac{2R}{\pi}\left(\underbrace{\sin\frac{\pi}{2}}_{1} - \underbrace{\sin 0}_{0}\right) \to \boxed{x_c = \frac{2R}{\pi}}$

β) $$z_c = \frac{2}{\pi \cancel{R}} \int_0^{\pi/2} R \sin\theta \, \cancel{R}\, d\theta = \frac{2R}{\pi} \int_0^{\pi/2} \sin\theta \, d\theta = \frac{2R}{\pi} \Big[-\cos\theta\Big]_0^{\pi/2} =$$

$$= \frac{2R}{\pi}\Big(-\underbrace{\cancel{\cos\tfrac{\pi}{2}}}_{0} + \underbrace{\cancel{\cos 0}}_{1}\Big) \longrightarrow \boxed{z_c = \frac{2R}{\pi}}$$



Ημικύκλιο συρμάτινο



όπου $\lambda = \frac{M}{\pi R}$

Ομοίως με άλλα όρια ολοκλήρωσης:

(α) $$x_c = \frac{R}{\pi} \int_0^{\pi} \cos\theta \, d\theta = \frac{R}{\pi}\Big(\underbrace{\cancel{\sin\pi}}_{0} - \underbrace{\cancel{\sin 0}}_{0}\Big) \longrightarrow \boxed{x_c = 0}$$

(β) $$z_c = \frac{R}{\pi} \int_0^{\pi} \sin\theta \, d\theta = \frac{R}{\pi}\Big(-\underbrace{\cancel{\cos\pi}}_{-1} + \underbrace{\cancel{\cos 0}}_{1}\Big) \longrightarrow \boxed{z_c = \frac{2R}{\pi}} \simeq 0{,}63R$$

Homework: να λυθεί αναλυτικά!!