Μεθοδικά, απλά & κατανοητά...

Θέμα

Σωματίδιο μάζας m βρίσκεται σε δυναμικό αρμονικού ταλαντωτή συχνότητας ω. Για $m = \hbar = \omega = 1$ θεωρήστε τον τελεστή

$$\hat{\alpha} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$$

(a) Είναι ο $\hat{\alpha}$ ερμητιανός;
(b) Υπολογίστε τον $\hat{\alpha}^\dagger$ και τον μεταθέτη $[\hat{\alpha}, \hat{\alpha}^\dagger]$.
(c) Γράψτε τον τελεστή της ενέργειας συναρτήσει των $\hat{\alpha}, \hat{\alpha}^\dagger$.
(d) Υπολογίστε την $\hat{\alpha}^\dagger\psi_0$, όπου $\psi_n, = 0, 1, 2, \dots$ οι ιδιοσυναρτήσεις της ενέργειας.
(e) Σχολιάστε ποια η σχέση της απάντησης του ερωτήματος (d) με την ψ_1.

α) Για να είναι ο $\hat{\alpha}$ ερμητιανός πρέπει να ισχύει: $\hat{\alpha}^\dagger = \hat{\alpha}$

Είναι: $\hat{\alpha}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})^\dagger = \frac{1}{\sqrt{2}}(\hat{x}^\dagger + i^*\hat{p}^\dagger) = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}) \neq \hat{\alpha}$

Δηλαδή ο $\hat{\alpha}$ δεν είναι ερμητιανός.

b) Είναι: $\hat{\alpha}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$ οπότε ο μεταθέτης των $\hat{\alpha}, \hat{\alpha}^\dagger$ είναι:

$$[\hat{\alpha}, \hat{\alpha}^\dagger] = \hat{\alpha}\hat{\alpha}^\dagger - \hat{\alpha}^\dagger\hat{\alpha} \quad (1)$$

όπου $\hat{\alpha}\hat{\alpha}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})\frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}) = \frac{1}{2}(\hat{x}^2 - i\hat{x}\hat{p} + i\hat{p}\hat{x} - \overset{-1}{\cancel{i^2}}\hat{p}^2) =$

$$= \frac{1}{2}(\hat{x}^2 + i(\hat{p}\hat{x} - \hat{x}\hat{p}) + \hat{p}^2) \rightarrow$$

$$\rightarrow \hat{\alpha}\hat{\alpha}^\dagger = \frac{1}{2}(\hat{x}^2 + i[\hat{p}, \hat{x}] + \hat{p}^2) \quad (2)$$

$\hat{\alpha}^\dagger\hat{\alpha} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})\frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) = \frac{1}{2}(\hat{x}^2 + i\hat{x}\hat{p} - i\hat{p}\hat{x} - \overset{-1}{\cancel{i^2}}\hat{p}^2) =$

$$= \frac{1}{2}(\hat{x}^2 - i(\hat{p}\hat{x} - \hat{x}\hat{p}) + \hat{p}^2) \rightarrow$$

$$\rightarrow \hat{\alpha}^{\dagger}\hat{\alpha} = \frac{1}{2}\left(\hat{x}^2 - i[\hat{p},\hat{x}] + \hat{p}^2\right) \quad (3)$$

Άρα η (1) λόγω των (2), (3) δίνει:

$$[\hat{\alpha},\hat{\alpha}^{\dagger}] = \frac{1}{2}\left(\cancel{\hat{x}^2} + i[\hat{p},\hat{x}] + \cancel{\hat{p}^2}\right) - \frac{1}{2}\left(\cancel{\hat{x}^2} - i[\hat{p},\hat{x}] + \cancel{\hat{p}^2}\right) = \frac{2i[\hat{p},\hat{x}]}{\cancel{2}} =$$

$$= i[\hat{p},\hat{x}] = i(-i\hbar) = -\underbrace{i^2}_{-1}\cancel{\hbar}$$

$\hbar = 1$ στο φυσικό σύστημα μονάδων.

$$\rightarrow \boxed{[\hat{\alpha},\hat{\alpha}^{\dagger}] = 1} \quad (4)$$

c) Ο τελεστής της ενέργειας είναι:

$$\hat{H} = K + V = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \xrightarrow{(m=\omega=1)} \hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} \quad (5)$$

Αλλά: $\hat{\alpha} + \hat{\alpha}^{\dagger} = \frac{2}{\sqrt{2}}\hat{x} \rightarrow \hat{x} = \frac{\sqrt{2}}{2}(\hat{\alpha} + \hat{\alpha}^{\dagger}) \rightarrow \hat{x} = \frac{1}{\sqrt{2}}(\hat{\alpha} + \hat{\alpha}^{\dagger}) \rightarrow$

$$\rightarrow \hat{x}^2 = \frac{1}{2}(\hat{\alpha} + \hat{\alpha}^{\dagger})(\hat{\alpha} + \hat{\alpha}^{\dagger}) \rightarrow \hat{x}^2 = \frac{1}{2}\left(\hat{\alpha}^2 + \hat{\alpha}\hat{\alpha}^{\dagger} + \hat{\alpha}^{\dagger}\hat{\alpha} + \hat{\alpha}^{\dagger 2}\right) \quad (6)$$

και $\hat{\alpha} - \hat{\alpha}^{\dagger} = \frac{2i}{\sqrt{2}}\hat{p} \rightarrow \hat{p} = \frac{\sqrt{2}}{2i}(\hat{\alpha} - \hat{\alpha}^{\dagger}) = \frac{i}{\sqrt{2}\underbrace{i^2}_{-1}}(\hat{\alpha} - \hat{\alpha}^{\dagger}) \rightarrow$

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$$\rightarrow \hat{p} = -\frac{i}{\sqrt{2}}(\hat{\alpha}-\hat{\alpha}^{+}) \rightarrow \hat{p} = \frac{i}{\sqrt{2}}(\hat{\alpha}^{+}-\hat{\alpha}) \rightarrow$$

$$\rightarrow \hat{p}^2 = \frac{i}{\sqrt{2}}(\hat{\alpha}^{+}-\hat{\alpha})\frac{i}{\sqrt{2}}(\hat{\alpha}^{+}-\hat{\alpha}) = \frac{i^2}{2}(\hat{\alpha}^{+2}-\hat{\alpha}^{+}\hat{\alpha}-\hat{\alpha}\hat{\alpha}^{+}+\hat{\alpha}^2) \rightarrow$$

$$\rightarrow \hat{p}^2 = -\frac{1}{2}(\hat{\alpha}^{+2}-\hat{\alpha}^{+}\hat{\alpha}-\hat{\alpha}\hat{\alpha}^{+}+\hat{\alpha}^2) \quad (7)$$

Οπότε η (5) λόγω των (6), (7) δίνει:

$$\hat{H} = -\frac{1}{4}(\hat{\alpha}^{+2}-\hat{\alpha}^{+}\hat{\alpha}-\hat{\alpha}\hat{\alpha}^{+}+\hat{\alpha}^2) + \frac{1}{4}(\hat{\alpha}^2+\hat{\alpha}\hat{\alpha}^{+}+\hat{\alpha}^{+}\hat{\alpha}+\hat{\alpha}^{+2}) =$$

$$= \frac{1}{4}(\cancel{-\hat{\alpha}^{+2}}+\hat{\alpha}^{+}\hat{\alpha}+\hat{\alpha}\hat{\alpha}^{+}\cancel{-\hat{\alpha}^2}\cancel{+\hat{\alpha}^2}+\hat{\alpha}\hat{\alpha}^{+}+\hat{\alpha}^{+}\hat{\alpha}\cancel{+\hat{\alpha}^{+2}}) = \rightarrow$$

$$= \frac{1}{4}(2\hat{\alpha}^{+}\hat{\alpha}+2\hat{\alpha}\hat{\alpha}^{+}) = \frac{2}{4}(\hat{\alpha}^{+}\hat{\alpha}+\hat{\alpha}\hat{\alpha}^{+}) \rightarrow$$

$$\rightarrow \hat{H} = \frac{1}{2}(\hat{\alpha}^{+}\hat{\alpha}+\hat{\alpha}\hat{\alpha}^{+}) \quad (8)$$

Αλλά από την (5): $[\hat{\alpha},\hat{\alpha}^{+}] = 1 \rightarrow \hat{\alpha}\hat{\alpha}^{+}-\hat{\alpha}^{+}\hat{\alpha} = 1 \rightarrow$

$$\rightarrow \hat{\alpha}\hat{\alpha}^{+} = 1+\hat{\alpha}^{+}\hat{\alpha} \quad (9)$$

Άρα η (8) λόγω της (9) δίνει:

$$\hat{H} = \frac{1}{2}(\hat{\alpha}^{+}\hat{\alpha}+1+\hat{\alpha}^{+}\hat{\alpha}) = \frac{1}{2}(2\hat{\alpha}^{+}\hat{\alpha}+1) \rightarrow$$

$$\rightarrow \boxed{\hat{H} = \hat{\alpha}^{+}\hat{\alpha}+\frac{1}{2}}$$, όπου $\hat{N} = \hat{\alpha}^{+}\hat{\alpha}$ ο τελεστής αριθμού κβάντων.

δ) Είναι: $\hat{a}^{\dagger}\psi_0 = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})\psi_0 = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})\,\pi^{-1/4} e^{-x^2/2} =$

$$= \frac{\pi^{-1/4}}{\sqrt{2}}\left[x e^{-x^2/2} - i\left(-i\overset{1}{\cancel{\hbar}}\frac{\partial}{\partial x} e^{-x^2/2}\right)\right] = \frac{\pi^{-1/4}}{\sqrt{2}}\left[x e^{-x^2/2} + \overset{-1}{\cancel{i^2}} e^{-x^2/2}\left(-\frac{\cancel{2}x}{\cancel{2}}\right)\right] =$$

$$= \frac{\pi^{-1/4}}{\sqrt{2}} \cdot 2x e^{-x^2/2} \rightarrow \boxed{\hat{a}^{\dagger}\psi_0 = \sqrt{2}\,\pi^{-1/4} x e^{-x^2/2}}$$

ε) Επειδή: $\psi_1 = \sqrt{2}\,\pi^{-1/4} x e^{-x^2/2}$ προκύπτει ότι: $\boxed{\hat{a}^{\dagger}\psi_0 = \psi_1}$

Δηλαδή επαληθεύεται η ιδιότητα του τελεστή $\hat{a}^{\dagger}$:

$$\hat{a}^{\dagger}\psi_n = \sqrt{n+1}\,\psi_{n+1}.$$