Εξίσωση συνέχειας στην κβαντομηχανική.

$$\frac{\partial P}{\partial t}+\frac{\partial J}{\partial x}=0.$$

όπου $P=\psi^*\psi$ η πυκνότητα πιθανότητας

και $J=\frac{\hbar}{2im}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right)$ η πυκνότητα του ρεύματος πιθανότητας

Θεωρούμε ότι το μονοδιάστατο δυναμικό $V(x)$ είναι πραγματική συνάρτηση οπότε:

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial t}(\psi^*\psi)=\psi^*\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi^*}{\partial t}. \quad (1)$$

Από τη χρονικά εξαρτημένη εξίσωση Schrödinger προκύπτουν:

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi \rightarrow \frac{\partial\psi}{\partial t}=\frac{1}{i\hbar}H\psi \quad (2)$$

και $\frac{\partial\psi^*}{\partial t}=\frac{1}{-i\hbar}H^*\psi^*$

όπου $H=\frac{p^2}{2m}+V \rightarrow H^*=\frac{p^{*2}}{2m}+V^*=\frac{p^2}{2m}+V=H$

$$p^2=pp=\left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)=i^2\hbar^2\frac{\partial^2}{\partial x^2}\Rightarrow p^2=-\hbar^2\frac{\partial^2}{\partial x^2}$$

[$i^2 = -1$]

$$\rightarrow \frac{\partial\psi^*}{\partial t}=-\frac{1}{i\hbar}H\psi^* \quad (3)$$

Οπότε η |1| λόγω των |2|, |3| είναι:

$$\frac{\partial \rho}{\partial t} = \psi^* \frac{1}{i\hbar} H\psi + \psi\left(-\frac{1}{i\hbar} H\psi^*\right) =$$

$$= \frac{1}{i\hbar}\left(\psi^* H\psi - \psi H\psi^*\right) \quad |4|$$

Αλλά: $H\psi = \left(\frac{p^2}{2m} + V\right)\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi \quad |5|$

και $H\psi^* = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^* \quad |6|$

Άρα η |4| λόγω των |5|, |6| είναι:

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\psi^* \frac{\partial^2 \psi}{\partial x^2} + V\psi^*\psi + \frac{\hbar^2}{2m}\psi\frac{\partial^2 \psi^*}{\partial x^2} - V\psi\psi^*\right)$$

$$\rightarrow \frac{\partial \rho}{\partial t} = \frac{-\hbar^2}{i\hbar 2m}\left(\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \psi^*}{\partial x^2}\right) =$$

$$= -\frac{\hbar}{2im}\frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right) = \frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x} + \psi^*\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x}\frac{\partial \psi^*}{\partial x} - \psi\frac{\partial^2 \psi^*}{\partial x^2}$$

$$\rightarrow \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\Big[\underbrace{\frac{\hbar}{2im}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right)}_{J}\Big] \rightarrow$$

$$\rightarrow \frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x} \rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0}$$ εξίσωση συνέχειας